INHOMOGENEOUS ELASTIC MEDIUM WITH NONLOCAL INTERACTION

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 3, pp. 60–66, 1967

In [1] the author examined a macroscopically homogeneous elastic medium of simple structure with spatial dispersion. In that case the assumption of the existence of an elementary unit of length and long-range forces conditioned the nonlocalizability of the theory, and the macroscopic homogeneity was manifested in the invariance of the integral operators under shear (difference kernels).

In this paper the more general model of an inhomogeneous elastic medium of simple structure with nonlocal interaction is constructed. In §1 the existence of a symmetric stress tensor is proved with broad assumptions, and the corresponding operator Hooke's law is written down. As a corollary, the usual expression for the energy density is obtained. In §2 the case of point defects is considered. An explicit expression is found for the Green's tensor for a medium with point defects in terms of the Green's tensor for the homogeneous medium. With the help of the Green's tensor the self-energy of the defect and the energy of the interaction force are calculated.

1. As in [1], we will assume that the state of the medium is completely determined by specifying the displacement field $u_\alpha(x)$ ($\alpha = 1, 2, 3$)(simple structure), where x is a point of the medium. We introduce the notation

$$\langle u \mid f \rangle = \int \overline{u(x)} f(x)\, dx = \frac{1}{(2\pi)^3} \int \overline{u(k)} f(k)\, dk = \overline{\langle f \mid u \rangle},$$

$$\langle u \mid \Phi \mid f \rangle = \iint \overline{u(x)}\, \Phi(x, x') f(x')\, dx\, dx' =$$

$$= \frac{1}{(2\pi)^3} \iint \overline{u(k)}\, \Phi(k, k') f(k')\, dk\, dk' = \overline{\langle f \mid \Phi^+ \mid u \rangle}, \quad (1.1)$$

where u, f, Φ must satisfy certain constraints of a function-theory nature for the corresponding functionals to be meaningful.

Here and in what follows u(k) denotes the Fourier transform of u(x), which is to be understood, if necessary, in the sense of generalized functions; $\Phi(k, k')$ is the Fourier transform of $\Phi(x, x')$ with respect to x and the Fourier original with respect to x'; $\Phi^+(k, k') = \overline{\Phi(k, k')}$, where the bar denotes complex conjugation.

In this notation the most general model of an inhomogeneous linear-elastic medium of simple structure is described by the Lagrangian

$$2L = \langle u_\alpha^{\cdot} \mid \rho g^{\alpha\beta} \mid u_\beta^{\cdot} \rangle - \langle u_\alpha \mid \Phi^{\alpha\beta} \mid u_\beta \rangle + 2 \langle u_\alpha \mid q^\alpha \rangle. \quad (1.2)$$

Here, $\rho(x)$ is the density, $g^{\alpha\beta}(x)$ the metric tensor, $q^\alpha(x)$ the external force density, and $\Phi^{\alpha\beta}(x, x')$ the kernel of the elastic energy operator, which may also be interpreted as the force constant matrix [1]. Obviously, in this case the Hermitian condition $\Phi = \Phi^+$ must be satisfied or, in more detailed form,

$$\Phi^{\alpha\beta}(x, x') = \Phi^{\beta\alpha}(x', x), \quad \Phi^{\alpha\beta}(k, k') = \overline{\Phi^{\beta\alpha}(k', k)}. \quad (1.3)$$

The corresponding equations of motion have the form

$$\rho(x) g^{\alpha\beta}(x) u_\beta^{\cdot\cdot}(x) - \int \Phi^{\alpha\beta}(x, x') u_\beta(x')\, dx' = q^\alpha(x). \quad (1.4)$$

In [1] it was shown that in a homogeneous medium of simple structure with nonlocal interaction it is possible to introduce a symmetric stress tensor. This conclusion is not completely trivial, since in many studies (see, for example, [2–5]) it has been asserted that if in a medium of simple structure the dependence of the energy not only on the strain but also on the strain gradient is taken into account, it is necessary to introduce an asymmetric stress tensor. (It can be shown that such a model corresponds to making an approximate allowance for nonlocalizability—weak spatial dispersion.) A similar problem arises in the case of an inhomogeneous medium with nonlocal interaction.

We will start by assuming that the medium is inhomogeneous only in a finite region. Then in a Cartesian coordinate system we have the unique representation

$$\Phi^{\alpha\beta}(x, x') = \Phi_0^{\alpha\beta}(x - x') + \Phi_1^{\alpha\beta}(x, x'),$$

$$\rho(x) = \rho_0 + \rho_1(x), \quad (1.5)$$

where $\Phi_0^{\alpha\beta}(x)$ and ρ_0 are the characteristics of the homogeneous medium, and the functions $\Phi_1^{\alpha\beta}(x, x')$ and $\rho_1(x)$ are finite.

Theorem. In an inhomogeneous medium of simple structure with nonlocal interaction, given the assumption of (1.5), we can introduce a symmetric stress tensor linked with the strain tensor by the operator Hooke's law, together with the energy density expressed as usual in terms of stress and strain.

Proof. From the conditions of invariance of the energy under translation and rotation it follows that $\Phi_0^{\alpha\beta}(k, k')$ can be represented [1] in the form

$$\Phi_0^{\alpha\beta}(k, k') = k_\nu k_\mu c_0^{\nu\alpha\mu\beta}(k)\, \delta(k - k'), \quad (1.6)$$

where $c_0^{\nu\alpha\mu\beta}(k)$ is a tensor symmetrical with respect to the first pair of indices, and $c_0^{\nu\alpha\mu\beta}(0)$ is the tensor of the elastic constants of the longwave approximation symmetrical with respect to the pairs $\nu\alpha$, $\mu\beta$ and their transposition. We will show that it is possible, without loss of generality, to assume that $c_0^{\nu\alpha\mu\beta}(k)$ is also symmetrical with respect to the second pair and Hermitian with respect to transposition of the pairs. (This possibility was not noticed in [1].)

We assume that this is not the case; then let

$$c_0^{\nu\alpha\mu\beta}(k) = c_0^{\nu\alpha(\mu\beta)}(k) + c_0^{\nu\alpha[\mu\beta]}(k). \quad (1.7)$$

Here, the parentheses and the brackets denote, as usual, symmetrization and antisymmetrization. Obviously, $c_0^{\nu\alpha[\mu\beta]}(0) = 0$ and, consequently, taking into account the analyticity of $c_0^{\nu\alpha\mu\beta}(k)$

$$c_0^{\nu\alpha[\mu\beta]}(k) = k_\lambda c_0^{\nu\alpha\mu\beta\lambda}(k), \quad (1.8)$$

where $c_0^{\nu\alpha\mu\beta}(k)$ is a tensor antisymmetric with respect to $\mu\beta$. The corresponding expression for the elastic energy Φ_0 can now be written in the form

$$2\Phi_0 = \langle u_\alpha | \Phi_0^{\alpha\beta} | u_\beta \rangle = \\ = \langle \varepsilon_{\nu\alpha} | c_0^{\nu\alpha(\mu\beta)} | \varepsilon_{\mu\beta} \rangle + \langle \varepsilon_{\nu\alpha} | c_0^{\nu\alpha\mu\beta\lambda} | k_\lambda \Omega_{\mu\beta} \rangle. \quad (1.9)$$

Here, $\varepsilon_{\nu\alpha}(k) = -ik_{(\nu}u_{\alpha)}(k)$ is the strain, $\Omega_{\mu\beta}(k) = -ik_{[\mu}u_{\beta]}(k)$ the rotation. The first term in the right side of (1.9) will not change if $c_0^{\nu\alpha(\mu\beta)}(k)$ is replaced by a tensor

$$a_1^{\nu\alpha\mu\beta} = \frac{1}{2}[c_0^{\nu\alpha(\mu\beta)} + \overline{c_0^{(\mu\beta)\nu\alpha}}] \quad (1.10)$$

having the necessary symmetry. Taking into account the identity $k_\lambda\Omega_{\mu\beta}(k) = 2k_{[\mu}\varepsilon_{\beta]\lambda}$, we rewrite the second term of (1.9) in the form

$$\langle \varepsilon_{\nu\alpha} | a_2^{\nu\alpha\mu\beta} | \varepsilon_{\mu\beta} \rangle ,$$

where

$$a_2^{\nu\alpha\mu\beta} = \frac{1}{2} k_\lambda (c_0^{\nu\alpha\lambda\mu\beta} + c_0^{\nu\alpha\lambda\beta\mu} + \overline{c_0^{\mu\beta\lambda\nu\alpha}} + \overline{c_0^{\beta\mu\lambda\nu\alpha}}) \quad (1.11)$$

also has the symmetry of $a_1^{\nu\alpha\mu\beta}$. Finally, combining the two terms, we obtain the required result.

We now turn to the term $\Phi_1^{\alpha\beta}(x, x')$ in (1.5). From the assumption that it is finite it follows that $\Phi_1^{\alpha\beta}(k, k')$ is an entire analytic function of k, k'. By means of reasoning similar to that conducted in [1], it is easy to show that the condition of invariance of the energy under translation is equivalent to the requirement that $\Phi_1^{\alpha\beta}(k, k')$ be representable in the form

$$\Phi_1^{\alpha\beta}(k, k') = k_\nu k_\mu' \psi^{\alpha\beta\nu\mu}(k, k'). \quad (1.12)$$

Here $\psi^{\alpha\beta\nu\mu}(k, k')$ is an entire analytic function which is uniquely determined by the assigned value of $\Phi_1^{\alpha\beta}(k, k')$ and satisfies the Hermitian conditions

$$\psi^{\alpha\beta\lambda\mu}(k, k') = \overline{\psi^{\beta\alpha\mu\lambda}(k', k)}. \quad (1.13)$$

Let the medium be subjected to the action of some equilibrated system of external forces and let $u_\beta(k)$ be the corresponding displacement. Then from the requirement of invariance of the energy under further rotation of the medium as a whole we find the condition on $\psi^{\alpha\beta\nu\mu}$ (the tensor $c_0^{\nu\alpha\mu\beta}$ satisfies this condition):

$$\mathrm{Re}\,\langle \Omega_{\nu\alpha}^* | \psi^{\alpha\beta\nu\mu} | k_\mu' u_\beta \rangle = 0. \quad (1.4)$$

Here, $\Omega_{\nu\alpha}{}^*(k) = a_{\nu\alpha}\,\delta(k)$ is the infinitesimal rotation given by the antisymmetric tensor $a_{\nu\alpha}$. It follows from (1.14) that the tensor

$$\psi_1^{\alpha\beta\nu\mu}(k') = \psi^{\alpha\beta\nu\mu}(0, k') \quad (1.15)$$

is symmetric with respect to the indices $\nu\alpha$.

We write $\psi^{\alpha\beta\nu\mu}(k, k')$ in the form

$$\psi^{\alpha\beta\nu\mu}(k, k') = \psi_1^{\alpha\beta\nu\mu}(k') + \psi_2^{\alpha\beta\nu\mu}(k, k'), \quad (1.16)$$

where, obviously, $\psi_2^{\alpha\beta\nu\mu}(0, k') = 0$. Consequently,

$$\psi_2^{\alpha\beta\nu\mu}(k, k') = k_\lambda \psi_2^{\alpha\beta\nu\lambda\mu}(k, k'). \quad (1.17)$$

In this case of the unique determination of $\psi_2^{\alpha\beta\nu\lambda\mu}(k, k')$ from a given $\Phi_1^{\alpha\beta}(k, k')$ it is necessary to require symmetry with respect to the indices $\nu\lambda$.

We will determine the tensor

$$c_1^{\nu\alpha\mu\beta}(k, k') = \psi_1^{\alpha\beta\nu\mu}(k') + k_\lambda[\psi_2^{\alpha\beta\nu\lambda\mu}(k, k') + \\ + \psi_2^{\nu\beta\alpha\lambda\mu}(k, k') - \psi_2^{\lambda\beta\alpha\nu\mu}(k, k')]. \quad (1.18)$$

From the properties of $\psi_1^{\alpha\beta\nu\mu}$ and $\psi_2^{\alpha\beta\nu\lambda\mu}$ there follows the symmetry of $c_1^{\nu\alpha\mu\beta}$ with respect to the first pair of indices.

The identity

$$\Phi_1^{\alpha\beta}(k, k') = k_\nu k_\mu' \psi^{\alpha\beta\nu\mu}(k, k') = k_\nu k_\mu' c_1^{\nu\alpha\mu\beta}(k, k') \quad (1.19)$$

is directly verified.

Calculations similar to those for $c_0^{\nu\alpha\mu\beta}$ show that $c_1^{\nu\alpha\mu\beta}$ in (1.19) can be replaced by a tensor having the symmetry of $c_0^{\nu\alpha\mu\beta}$. Retaining the same notation $c_1^{\nu\alpha\mu\beta}$, we write

$$c^{\nu\alpha\mu\beta}(x, x') = c_0^{\nu\alpha\mu\beta}(x - x') + c_1^{\nu\alpha\mu\beta}(x, x'). \quad (1.20)$$

For the elastic energy Φ we now have

$$2\Phi = \langle \varepsilon_{\nu\alpha} | c^{\nu\alpha\mu\beta} | \varepsilon_{\mu\beta} \rangle. \quad (1.21)$$

Hence it follows that the equations of motion are written in the form

$$\rho(x)\,\delta^{\alpha\beta}\ddot{u}_\beta(x) - \partial_\nu \sigma^{\nu\alpha}(x) = q^\alpha(x), \quad (1.22)$$

where

$$\sigma^{\nu\alpha}(x) = \int c^{\nu\alpha\mu\beta}(x, x')\,\varepsilon_{\mu\beta}(x')\,dx'. \quad (1.23)$$

Obviously, the symmetric tensor $\sigma^{\nu\alpha}$ can be interpreted as the stress tensor, and relation (1.23) as the operator Hooke's law.

Finally, from (1.21), using (1.23), we find that the quantity

$$\varphi(x) = {}^1/_2\,\sigma^{\nu\alpha}(x)\,\varepsilon_{\nu\alpha}(x), \quad (1.24)$$

which is invariant under rotation of the medium as a whole, is the elastic energy density. This proves the theorem.

Remarks. 1. Expression (1.24) shows that in this case the stress tensor, in complete correspondence with the local theory of elasticity, can also be defined as the generalized force corresponding to the generalized displacement—the strain tensor.

2. It follows from the proof that the finiteness of $\Phi_1^{\alpha\beta}(x, x')$ is not a necessary condition. It can be weakened by assuming only the analyticity of $\Phi_1^{\alpha\beta}(k, k')$.

3. The fact that the displacement is the only function determining the state of the medium was put to important use in the proof, therefore the latter does not extend to cases of internal stresses and media of complex structure.

2. We will consider the important practical model of point defects in a homogeneous medium with spatial dispersion.

It is assumed that an elementary unit of length a exists in the medium. This is equivalent to the assumption that the Fourier transforms of admissible functions are nonzero only in a certain finite neighborhood B of the coordinate origin of k-space [1]. Thus, for a regular lattice the region B is a three-dimensional torus, and for an isotropic medium (Debye model) a sphere. In all cases the characteristic dimension of the region B is of the order of a^{-1}.

We note one important consequence of the existence of an elementary unit of length. In the case of admissible functions the role of the δ-function is played by the regular function $\delta_B(x)$, whose specific form depends only on the region B. For example, for an isotropic model ($r = |x|$, $\varkappa = \pi a^{-1}$)

$$\delta_B(x) = \frac{\varkappa}{2\pi^2 r^2}\left(\frac{\sin \varkappa r}{\varkappa r} - \cos \varkappa r\right) \tag{2.1}$$

and the expression for $\delta_B(x)$ for a regular lattice is given in [1]. When $a \to 0$ the δ-function coincides with the usual form. In what follows the subscript B for the δ-function will be omitted.

We will start by assuming that a single defect exists in the medium at the point $x = x_0$, and for the time being we will confine ourselves to the case of statics. Then the simplest model of a point defect can be represented in the form ($v = a^3$)

$$c^{\nu\alpha\mu\beta}(x, x') = c_0^{\nu\alpha\mu\beta}(x - x') + $$
$$+ v c_1^{\nu\alpha\mu\beta}\,\delta(x - x_0)\,\delta(x' - x_0). \tag{2.2}$$

Here, the tensor $c_1^{\nu\alpha\mu\beta}$ has the dimension of a modulus of elasticity and characterizes the change in the elastic properties of the medium in the neighborhood of the defect. Comparison with (1.13) shows that correct to the multiplier $v\,c_1^{\nu\alpha\mu\beta}$ coincides with $\psi_2^{\alpha\beta\nu\mu}(0)$, and $\psi_2^{\alpha\beta\lambda\nu\mu} = 0$. Considering the symmetry of $\psi_1^{\alpha\beta\nu\mu}$ and relation (1.8), we conclude that $c_1^{\nu\alpha\mu\beta}$ is symmetric with respect to the indices of the first and second pairs and with respect to transposition of the pairs, i.e., has the symmetry of the usual elastic constant tensor. We note that a more complex model of a point defect is obtained if we include terms with derivatives of the δ-function in (2.2).

Assuming that the Green's tensor $\overset{\circ}{G}_{\alpha\beta}(x - x')$ for a homogeneous medium is known, we will construct the Green's tensor $G_{\alpha\beta}(x, x')$ for a medium with a defect. In the case of an isotropic medium and, in particular, for a Debye model the expression $\overset{\circ}{G}_{\alpha\beta}(x)$ can be written in explicit form.

From (1.15), using (2.2) and (1.16), we find the equation for the static Green's tensor

$$\partial_\nu \int c_0^{\nu\lambda\mu\alpha}(x - y)\,\partial_\mu G_{\alpha\beta}(y, x')\,dy + $$
$$+ v c_1^{\nu\lambda\mu\alpha}\,\partial_\nu \delta(x - x_0)\,\partial_\mu G_{\alpha\beta}(x_0, x') = -\delta_\beta^\lambda \delta(x - x'). \tag{2.3}$$

Applying $\overset{\circ}{G}_{\alpha\beta}(x - x')$ to both sides gives

$$G_{\alpha\beta}(x, x') - v c_1^{\nu\mu\tau\lambda}\partial_\nu \overset{\circ}{G}_{\alpha\mu}(x - x_0)\,\partial_{(\tau}G_{\lambda)\beta}(x_0, x') = $$
$$= \overset{\circ}{G}_{\alpha\beta}(x - x'). \tag{2.4}$$

This expression contains the unknown quantity $\partial_{(\tau}G_{\lambda)\beta}(x_0, x')$. In order to find it, we take the symmetricized gradient of both sides of the equation and set $x = x_0$. We obtain the algebraic equation

$$[\delta_{(\varkappa}{}^{\tau}\delta_{\alpha)}{}^{\lambda} + v g_{\varkappa\alpha\nu\mu} c_1^{\nu\mu\tau\lambda}]\,\partial_{(\tau}G_{\lambda)\beta}(x_0, x') = $$
$$= \partial_{(\varkappa}\overset{\circ}{G}_{\alpha)\beta}(x_0 - x'), \tag{2.5}$$

where

$$\overset{\circ}{g}_{\varkappa\alpha\nu\mu} = -[\partial_\varkappa \partial_\nu \overset{\circ}{G}_{\alpha\mu}(0)]_{(\varkappa\alpha)(\nu\mu)}. \tag{2.6}$$

Solving Eq. (2.5) and substituting the result into (2.4), we write the final expression for $G_{\alpha\beta}(x, x')$ in the form

$$G_{\alpha\beta}(x, x') = \overset{\circ}{G}_{\alpha\beta}(x - x') + $$
$$+ \partial_\nu \overset{\circ}{G}_{\alpha\mu}(x - x_0)\,P^{\nu\mu\lambda\tau}\partial_\lambda \overset{\circ}{G}_{\beta\tau}(x_0 - x'). \tag{2.7}$$

Here, the constant tensor $P^{\nu\mu\lambda\tau}$ has the symmetry of the elastic constant tensor and is given by

$$P^{\nu\mu\lambda\tau} = (\overset{\circ}{g}_{\nu\mu\lambda\tau} + v^{-1} b_{\nu\mu\lambda\tau})^{-1}. \tag{2.8}$$

The tensor $b_{\nu\mu\lambda\tau}$ is the inverse of $c_1^{\nu\mu\lambda\tau}$.

In deriving the expression for $G_{\alpha\beta}(x, x')$ we implicitly assumed the existence of an elementary unit of length. As a result of this assumption $\overset{\circ}{G}_{\alpha\beta}(x)$ is an entire analytic function [1] and the derivatives at zero are finite. It follows from (2.7) that $G_{\alpha\beta}(x, x')$ is also an entire function of x, x'. At the same time, $G_{\alpha\beta}(x, x')$ depends nonanalytically on the parameter a. It is easy to show that $\nabla \overset{\circ}{G}(0) \sim a^{-2}$, $\overset{\circ}{g} \sim v^{-1}$ and, consequently, $P \sim v$. As $a \to 0$ the second term in (2.7) tends to zero as a^3 for the points $x, x' \neq x_0$, and as a for the points $x = x_0$, $x' \neq x_0$ or $x \neq x_0$, $x' = x_0$, and tends to infinity as a^{-1} for the point $x = x_0$, $x' = x_0$.

The dynamic Green's tensor for a point defect can be similarly constructed. In this instance it is desirable to consider the two cases separately. If the density is constant and the defect is due only to a change in the elastic properties of the medium, the expression for the dynamic Green's tensor $G_{\alpha\beta}(x, x', \omega)$, where ω is the frequency, coincides with (2.7) with $\overset{\circ}{G}_{\alpha\beta}(x)$ replaced by the dynamic Green's tensor of the homogeneous medium $\overset{\circ}{G}_{\alpha\beta}(x, \omega)$. Correspondingly, P will be a function of ω determined by the relation (2.8).

In the other case, when $c(x, x') = c_0(x - x')$, and the density has the form

$$\rho(x) = \rho_0 + v\rho_1 \delta(x - x_0), \tag{2.9}$$

we find for the Green's tensor

$$G_{\alpha\beta}(x, x', \omega) = \overset{\circ}{G}_{\alpha\beta}(x - x', \omega) + $$
$$+ \overset{\circ}{G}_{\alpha\nu}(x - x_0, \omega)\,P^{\nu\mu}(\omega)\,\overset{\circ}{G}_{\mu\beta}(x_0 - x', \omega), \tag{2.10}$$

where

$$P^{\nu\mu}(\omega) = [(v\rho_1\omega^2)^{-1}\delta_{\nu\mu} - \overset{\circ}{G}_{\nu\mu}(0, \omega)]^{-1}. \tag{2.11}$$

The expression for the Green's tensor in the general case, when there is both a mass defect and an elastic modulus defect, is rather clumsy and therefore has been omitted.

We will now consider the case of several defects—to be specific, defects of the elastic moduli. For simplicity, moreover, we will confine ourselves to writing the static Green's tensor. Let

$$c^{\nu\alpha\mu\beta}(x, x') = c_0^{\nu\alpha\mu\beta}(x - x') + v \sum_i c_i^{\nu\alpha\mu\beta} \delta(x - x_i) \delta(x' - x_i). \quad (2.12)$$

Omitting the straightforward, but cumbersome calculations, we give the final result

$$G_{\alpha\beta}(x, x') = \overset{\circ}{G}_{\alpha\beta}(x - x') + \sum_{ij} \partial_\nu \overset{\circ}{G}_{\alpha\mu}(x - x_i) P_{ij}^{\nu\mu\lambda\tau} \partial_\lambda \overset{\circ}{G}_{\beta\tau}(x_j - x'). \quad (2.13)$$

Here, the matrix P_{ij} is the inverse of the matrix

$$R_{\nu\mu\lambda\tau}^{ij} = \overset{\circ}{g}{}_{\nu\mu\lambda\tau}^{ij} + v^{-1}\delta^{ij} b_{\nu\mu\lambda\tau}^{j}, \quad (2.14)$$

where

$$\overset{\circ}{g}{}_{\nu\mu\lambda\tau}^{ij} = -[\partial_\nu \partial_\lambda \overset{\circ}{G}_{\mu\tau}(x_i - x_j)]_{(\nu\mu)(\lambda\tau)}, \quad (2.15)$$

and the tensors $b^j_{\nu\mu\lambda\tau}$ are the inverse of $c_j^{\nu\mu\lambda\tau}$.

For the important case of two defects (i, j = 1, 2), the expressions for calculating the components P_{ij} can be substantially simplified by representing them in the form $(i \neq j)$

$$P_{ii} = (R_{ii} - R_{ij} R_{jj}^{-1} R_{ji})^{-1},$$

$$P_{ij} = (R_{ji} - R_{jj} R_{ij}^{-1} R_{ii})^{-1}. \quad (2.16)$$

The Green's tensor for the case of mass defects has a similar structure.

In conclusion, we present the equations for the energy and interaction forces of the defects, which are explicitly expressed in terms of the static Green's tensor.

The energy Φ of an arbitrary system of external forces of density $q^\alpha(x)$ can be written in the form

$$2\Phi = \langle \varepsilon_{\nu\beta} | \sigma^{\nu\beta} \rangle = \langle u_\beta | q^\beta \rangle = \langle q^\alpha | G_{\alpha\beta} | q^\beta \rangle. \quad (2.17)$$

Let a defect at the point x_0 be associated with a force dipole of density

$$q^\alpha(x) = -Q^{\nu\alpha} \partial_\nu \delta(x - x_0). \quad (2.18)$$

Here, $Q^{\nu\alpha} = Q^{\alpha\nu}$ is the dipole moment. Then for the self-energy of the defect from (2.17) we find

$$2\Phi = Q^{\nu\alpha} Q^{\mu\beta} g_{\nu\alpha\mu\beta}, \quad (2.19)$$

where

$$g_{\nu\alpha\mu\beta} = [\partial_\nu \partial_\mu' G_{\alpha\beta}(x_0, x_0)]_{(\nu\alpha)(\mu\beta)}. \quad (2.20)$$

From (2.19) there follows, in particular, the physical significance of the quantity $g^\circ_{\nu\alpha\mu\beta}$ in (2.8).

In the case of a system of defects the total energy is written in the form

$$2\Phi = \sum_{ij} Q_i^{\nu\alpha} Q_j^{\mu\beta} g_{\nu\alpha\mu\beta}^{ij}, \quad (2.21)$$

where

$$g_{\nu\alpha\mu\beta}^{ij} = [\partial_\nu \partial_\mu' G_{\alpha\beta}(x_i, x_j)]_{(\nu\alpha)(\mu\beta)}. \quad (2.22)$$

We note that in the sum in (2.21) the terms with i = j cannot be identified with the self-energy of the defect, since in this case the Green's tensor is determined by the totality of the defects.

The force acting on a defect at the point $x = x_k$ due to the other defects is defined as

$$f_\lambda^k = -\frac{\partial}{\partial x_k^\lambda} \Phi. \quad (2.23)$$

Calculations give

$$f_\lambda^k = \sum_{ij} Q_i^{\nu\alpha} Q_j^{\mu\beta} [\delta^{ki} \partial_\lambda \partial_\nu \partial_\mu' G_{\alpha\beta}(x_i, x_j) + G_{\lambda\alpha\beta}^k(x_i, x_j)], \quad (2.24)$$

where

$$G_{\lambda\alpha\beta}^k(x, x') = -\frac{1}{2} \frac{\partial}{\partial x_k^\lambda} G_{\alpha\beta}(x, x'). \quad (2.25)$$

REFERENCES

1. I. A. Kunin, "Model of an elastic medium of simple structure with spatial dispersion," PMM, vol. 30, no. 3, p. 542, 1966.

2. E. L. Aero and E. V. Kuvshinskii, "Basic equations of the theory of elasticity with rotational particle interaction," Fiz. tverdogo tela, vol. 2, no. 7, p. 1399, 1960.

3. G. Grioli, "Elasticita asimmetrica," Ann. mat. Pura ed Appl., Ser. IV, vol. 50, p. 382, 1960.

4. R. A. Toupin, "Elastic materials with couple-stresses," Arch. Rat. Mech. Anal., vol. 11, no. 5, p. 385, 1962.

5. R. D. Mindlin and H. F. Tiersten, "Effects of couple-stresses in linear elasticity," Arch. Rat. Mech. Anal., vol. 11, no. 5, p. 415, 1962.

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